PG – 114

# IV Semester M.Sc. Degree Examination, June 2015
## (RNS)
## MATHEMATICS
## M-401 : Measure and Integration

Time : 3 Hours

Max. Marks : 80

***Instruction :*** *1) Answer **any 5** questions, choosing atleast **two** from **each** Part.*

*2) **All** questions carry **equal** marks.*

PART – A

1. a) Define outer measure. Prove that outer measure is translation invariant.

   b) For an interval I, show that $m^*(I) = l(I)$. Further, if A is a countable set, prove that $m^*(A) = 0$

   c) Define a measurable set. If A is measurable set and B is any set, then show that $m^*(A \cup B) + m^*(A \cap B) = m^*(A) + m^*(B)$. **(4+8+4)**

2. a) State and prove countably additive property of Lebesgue measurable sets.

   b) Let A be any subset of R. If $E_1, E_2, \ldots, E_n$ are disjoint Lebesgue measurable sets, then prove that $m^*\left(A \cap \left(\bigcup_{i=1}^{n} E_i\right)\right) = \sum_{i=1}^{n} m^*(A \cap E_i)$.

   c) Prove that a set A is measurable if and only if its complement is also measurable. **(6+7+3)**

3. a) Prove that every Borel set is Lebesgue measurable.

   b) If f and g are two measurable real valued functions defined on the same domain, then prove that f-g, cf and $f^2$ are also measurable.

   c) Define a measurable function. Show that the following statements are equivalent for a function $f : E \to R^*$, where $R^*$ denotes the extended real number system

      i) $\{x \in E / f(x) > a\}$ is measurable $\forall a \in R$

      ii) $\{x \in E / f(x) \geq a\}$ is measurable $\forall a \in R$

      iii) $\{x \in E / f(x) < a\}$ is measurable $\forall a \in R$

P.T.O.

iv) $\{x \in E / f(x) \le a\}$ is measurable $\forall a \in R$

Further, show that the above statement imply that for any $b \in R^*$, $\{x \in E / f(x) = b\}$ is measurable. **(6+2+8)**

4. a) Let E be a Lebesgue measurable set with finite measure, for a given $\in > 0$, prove that there exists a finite union 'U' of open intervals such that $m(E \Delta U) < \varepsilon$, where $E \Delta U = (E - U) \cup (U - E)$.

   b) Let f be a measurable function and g be a function defined over a measurable set E, such that f = g a.e. on E. Then prove that g is measurable.

   c) If a sequence $\{f_n\}$ converges on measure to f, then prove the following :

      i) $\{f_n\}$ converges on measure to every function g which is equivalent to f.

      ii) The limit function f is unique a. e.. **(7+4+5)**

PART – B

5. a) If f is Lebesgue integral over E, then show that $\alpha mE \le \int_E f \le \beta mE$.

   b) Let f be a bounded function on E with $mE < \infty$. Then show that f is measurable iff $\inf_{\psi \ge f} \int_E \psi = \sup_{\phi \le f} \int \phi$ for all simple functions $\phi$ and $\psi$.

   c) Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions defined on a set E of finite measure. Suppose $|f_n(x)| \le M \ \forall n$ and $\forall x \in E$. If $\lim_{n \to \infty} f_n(x) = f$ for each $x \in E$, then show that $\lim_{n \to \infty} \int_E f_n = \int_E f$. **(2+10+4)**

6. a) State and prove Fatou's lemma. Deduce Monotone Convergence Theorem. Further, give an example to show that montone convergence theorem need not hold for a decreasing sequence of functions.

   b) Let f be a non-negative measurable function which is integrable over a set E. Then prove that for a given $\varepsilon > 0$ there is a $\delta > 0$ such that for every set $A \subset E$ with $mA < \delta$ are have $\int_A f < \varepsilon$. **(10+6)**